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On the reattachment of a shock layer produced by an instantaneous energy release

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The behaviour of a strong shock wave, which is initiated by a point explosion and driven continuously outward by an inner contact surface (or a piston), is studied as a problem of multiple time scales for an infinite shock strength, $\dot{y}_{sh}/a_\infty \rightarrow \infty$, and a high shock-compression ratio, $\rho_s/\rho_\infty \sim 2\gamma/(\gamma-1) \equiv \epsilon^{-1} \gg 1$. The asymptotic analyses are carried out for cases with planar and cylindrical symmetry in which the piston velocity is a step function of time. The solution shows that the transition from an explosion-controlled régime to that of a reattached shock layer is characterized by an oscillation with slowly-varying frequency and amplitude. In the interval of a scaled time $1 \ll t \ll \epsilon^{-2/3(1+\nu)}$, the oscillation frequency is shown to be $(1+\nu)(2\pi)^{-1}t^{-\frac{1}{2}(1-\nu)}$ and the amplitude varies as $t^{-\frac{1}{4}(3+\nu)}$, matching the earlier results of Cheng *et al.* (1961). The approach to the large-time limit, $\epsilon^{1/(1+\nu)}t \rightarrow \infty$, is found to involve an oscillation with a much reduced frequency, $\frac{1}{4}\pi(1+\nu)\epsilon^{-\frac{1}{2}}t^{-1}$, and with an amplitude decaying more rapidly like $\epsilon^{-\frac{5}{4}}t^{-\frac{1}{2}(4+3\nu)}$; this terminal behaviour agrees with the fundamental mode of a shock/acoustic-wave interaction.

1. Introduction

The dynamics and structure of a blast wave, initiated by a point explosion and driven outward continuously by an inner contact surface, has been analysed recently by Cheng & Kirsch (1969). The analysis, which is asymptotic for a high shock-compression ratio, treats the interaction of a shock layer and an inner region, called the entropy wake of explosion. For a monotonic contact-surface motion which is faster than the shock motion of a pure (Taylor–Sedov) blast wave, the analysis shows that the shock layer will be caught up by the surface at large time, that is, the shock layer will 'reattach' to the surface, and that a decaying oscillation generally accompanies the reattachment. However, close examination reveals that the higher-order terms in Cheng & Kirsch's (1969) expression cease to be valid at a large time. In the present study, we treat the large-time oscillatory problem as one with multiple time scales and analyse in detail the transition to reattachment for the plane and cylindrical cases in which the piston velocity is a step function of time, as illustrated in figure 1. The corresponding problems with spherical symmetry and with initial density stratification are treated in subsequent work (Moh 1971).

Similar oscillatory behaviour was identified earlier by Cheng *et al.* (1961), in the analysis of the equivalent problems of hypersonic flows past blunted wedges and cones using a rather crude Newtonian model. Although an oscillation is

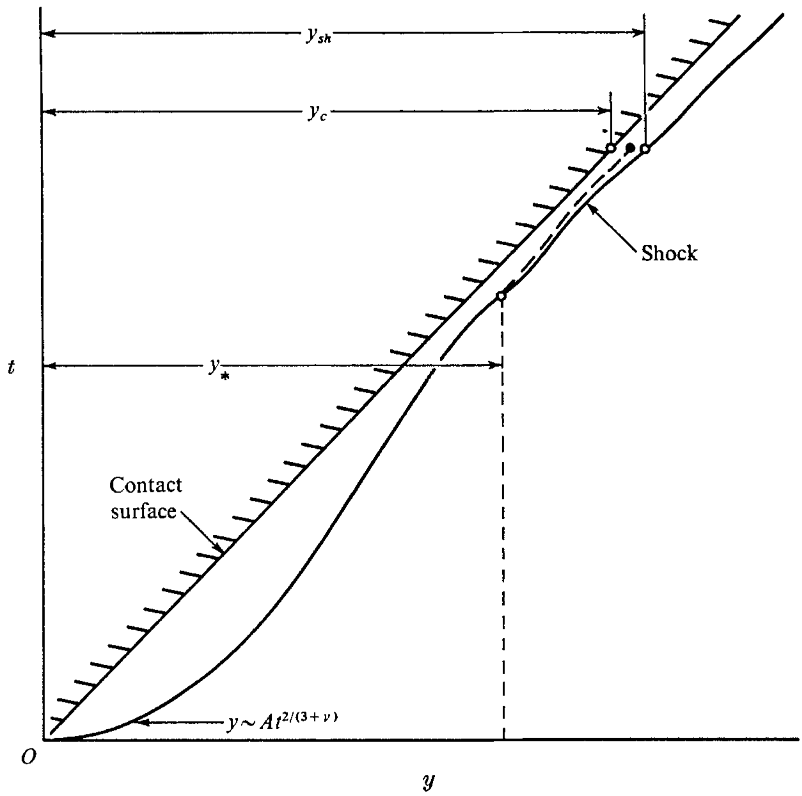

FIGURE 1. A wave diagram showing a blast wave driven outward by a contact surface expanding from the origin of the blast. In the sketch, t is the time and y is the distance from the spatial origin of the blast.

not apparent in the well-known work of Chernyi (1959) (see, for example, Mirels 1962, Cox & Crabtree 1965, Guiraud *et al.* 1965, and in particular, pp. 363–364 of Hayes & Probstein 1966) a distinct wavy pattern can be identified by both asymptotic and numerical analyses as noted recently by Schneider (1968). In a study of the downstream asymptotic behaviour of inviscid hypersonic flows, to be discussed more specifically later in §7, Ellinwood (1967) reports that, for a blunted wedge or cone, a whole family of eigensolutions, each with a distinct characteristic frequency, is admissible to his model flow. However, the result cannot be considered as evidence of oscillation in solutions to problems with prescribed piston motions, since the eigensolutions are infinite in number. The main objectives of the present analysis are to arrive at a Newtonian solution valid at large (and all) time, and to clarify the issue on the oscillation.

From the theoretical gas-dynamics viewpoint, the problem under study is of interest in that it furnishes for the first time examples wherein the history of reattachment of a Newtonian shock layer can be analytically delineated. The analysis is also of interest as an example in the method of multiple time scales applied to *partial* differential equations, not treated in standard works (Van Dyke 1964; Cole 1968, ch. 3). A hydrodynamic model involving an explosion in a spherically stratified atmosphere has been proposed by Parker (1963, pp. 92–112) for the sudden solar corona expansion (see Goldworthy 1969). A similar oscillation in Parker's model may be anticipated at large time, if the model remains valid. Wavy shock patterns at large time or distance can develop even in cases in which the contact-surface motion is not prescribed *a priori*, such as being found in a study of the combined bluntness and boundary-layer displacement effect in an axisymmetric hypersonic flow (Cheng 1969), in which the contact surface is replaced by the boundary-layer outer edge.†

To conserve space without sacrificing clarity, most analytical details are presented for the piston problem in the planar case. The corresponding solution to the cylindrical case can only be sketched briefly. In the next section, the nature of the large-time problem is described; the basic construction and the proper variables for the principal transition region are introduced and studied in §3; the analysis of the piston problem in the planar case is presented in §§4 and 5; the corresponding analysis of the cylindrical piston problem is summarized in §6; and the final results of the two analyses are discussed in §7.

2. Basic equations and nature of the large-time problem

2.1. *The exact model problem*

Following the basic formulation of Cheng & Kirsch (1969), it is assumed that there is one shock of infinite strength separating the disturbed and the undisturbed uniform regions; that the fluid motion is (inviscid) particle-isentropic; that the gas is calorically perfect; and that immediately after the explosion, the field is described by the constant-energy solution.

The differential equations governing the model explosion problem, specialized to spatially symmetric motion in one, two, and three dimensions, can be written as

$$\rho_\infty \frac{\partial^2 y}{\partial t^2} = -\left(\frac{y}{y_*}\right)^\nu \frac{\partial p}{\partial y_*}, \quad \frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right) = 0, \quad \left(\frac{y}{y_*}\right)^\nu \frac{\partial y}{\partial y_*} = \frac{\rho_\infty}{\rho}, \quad v = \frac{\partial y}{\partial t}, \tag{2.1}$$

where y, p, ρ, v and γ are distance from the blast centre, pressure, density, velocity, and specific-heat ratio of the gas, respectively; the index ν takes on

† Flows past concave (as well as certain convex) slender bodies in the hypersonic strong-interaction régime also exhibit similar behaviours under the Newtonian approximation, even in the absence of an entropy wake and/or entropy layer. The latter's role is taken up in this case by the low-density hypersonic boundary layer. An example of this kind is noted recently by Stollery (1970), to which a two-timing analysis is also applicable.

0, 1 and 2 for planar, cylindrical and spherical symmetry, and the subscript ∞ refers to the uniform initial state. The independent variables t and y_* are, respectively, the time and a Lagrangian co-ordinate. The trajectory of a particle in the physical t, y plane may then be described by $y(t, y_*)$ with y_* signifying the particle ordinate y when the particle crosses the shock (see figure 1). The outer boundary conditions are furnished by the Rankine–Hugoniot conditions which for an infinite shock strength and with the shock ordinate written as $y = y_{sh}(t)$, are

$$p = \frac{2}{\gamma+1}\rho_\infty \dot{y}_{sh}^2, \quad \rho = \frac{\gamma+1}{\gamma-1}\rho_\infty, \quad y = y_{sh}(t) = y_*, \tag{2.2}$$

where the dot stands for the time derivative.

The boundary condition at the inner contact surface is

$$y = y_c(t) \quad \text{at} \quad y_* = 0 \tag{2.3}$$

if the motion of the contact surface is prescribed. In the case analysed in detail below, $y_c \propto t$ for $t > 0$.

The system (2.1)–(2.3) admits a particular solution which approaches the self-similar constant-energy solution in the limit $t \to 0$, i.e.

$$y_{sh} \sim At^{2/(3+\nu)}, \quad \text{as} \quad t \to 0, \tag{2.4}$$

where A is a constant of integration, provided $y_c/t^{2/(3+\nu)} \to 0$ as $t \to 0$. The specification of the initial energy release E_0 determines A and the particular solution desired for $t > 0$.

2.2. *The Newtonian theory and non-uniformity*

The analysis of Cheng & Kirsch (1969) made under

$$\epsilon \equiv (\gamma-1)/2\gamma \leqslant 1$$

dealt primarily with a régime in which the field between the shock and the piston is occupied *in most part* by a highly stratified low-density region – the 'entropy wake'. This régime, or period, may be referred to as the 'explosion-controlled régime' in that the initial energy release dominates the energy balance of the flow (although the field cannot be generally described by the self similar constant-energy solution). Matching of the solutions yields a relation between the surface pressure and the shock co-ordinate, called 'pressure-volume relation', in each order of the approximations. With the pressure furnished by the Busemann formula for the leading order, and a similar formula for the next, the relation leads to an ordinary differential equation for each coefficient in the shock-co-ordinate expansion

$$Y \equiv y_c/b = Y_0(\hat{t}) + \epsilon Y_1(\hat{t}) + \epsilon^2 Y_2(\hat{t}) + \dots, \tag{2.5}$$

where $\hat{t} \equiv t/\tau$, and b and τ are properly chosen length and time scales, respectively. Underlying the analysis is the use of a time scale (with $k_\nu = 1$, 2π and 4π for $\nu = 0$, 1 and 2, respectively)

$$\tau = [\rho_\infty k_\nu b^{(3+\nu)}/2(1+\nu)\epsilon E_0]^{\frac{1}{2}}, \tag{2.6}$$

which ensures that all dimensionless variables including Y belong to the unit order in the explosion-controlled régime. For obvious reasons, the régime $\hat{t} = O(1)$ will be alternatively referred to as the 'finite-time period'. In subsequent analysis, the cap in the symbol $\hat{t}$ will be omitted for convenience.

For the special case $y_c \propto t$, we may set $b/\tau = \dot{y}_c$; the contact surface may then be written as $Y_c \equiv y_c/b = t$. For this case, the finite-time analysis (Cheng & Kirsch 1969) gives the oscillatory behaviour at large t

$$\hat{p}_c \equiv \frac{p_c}{\rho_\infty \dot{y}_c^2} \sim 1 - \frac{(1+\nu)^2 A_0}{t^{\frac{1}{4}(3+\nu)}} \cos\left[2t^{\frac{1}{2}(1+\nu)} + \phi_0\right] - \epsilon\{B_0 t^{\frac{1}{4}(3+5\nu)} \sin\left[2t^{\frac{1}{2}(1+\nu)} + \phi_0\right] + \ldots\} + O[\epsilon^2 t^{\frac{1}{4}(9+11\nu)}], \quad (2.7)$$

where B_0 is $\frac{1}{9}(1+\nu)^2 A_0$, and the constants A_0 and ϕ_0 can be quite accurately inferred from numerical integrations in the leading approximations (Kirsch 1969), giving $A_0 \simeq 0{\cdot}72$, $\phi \simeq 79°$ for $\nu = 0$ and $A_0 \simeq -1{\cdot}08$ and $\phi = 76°$ for $\nu = 1$. The corresponding behaviour for Y may be inferred from the pressure-volume relation. Clearly from (2.7), the finite-time analysis cannot give a uniformly valid expansion in ϵ for an unlimited t.

2.3. *Two transitions to reattachment*

Equation (2.7) reveals readily that a non-uniformity of the finite-time analysis occurs in the period $t = O[\epsilon^{-2/3(1+\nu)}]$. An asymptotic solution for this period, in turn, leads to a breakdown in a later period $t = O[\epsilon^{-1/(1+\nu)}]$, as will be evident from subsequent analyses. For reasons to become obvious later, the first period of non-uniformity will be designated as the 'incipient transition period', and the second as the 'principal transition period'. These two non-uniformities may be seen as the consequence of a frequency modulation—a weak time dependence of the oscillation frequency. This can best be illustrated by a closer examination of (2.7), which will also clearly bring out the motivation for introducing the more appropriate scalings.

Under the stipulation that the non-uniformity arises primarily from a frequency modulation, (2.7) may be *interpreted* as either of

$$\hat{p}_c - 1 \sim -(1+\nu)^2 A_0 t^{-\frac{1}{4}(3+\nu)} \cos\{2t^{\frac{1}{2}(1+\nu)} + \phi_0 - \tfrac{1}{9}\epsilon t^{\frac{3}{2}(1+\nu)}\}, \quad (2.8a)$$

$$\hat{p}_c - 1 \sim -(1+\nu)^2 A_0 t^{-\frac{1}{4}(3+\nu)} \cos\{[2 - \tfrac{1}{9}\epsilon t^{(1+\nu)}]\, t^{\frac{1}{2}(1+\nu)} + \phi_0\}. \quad (2.8b)$$

It is apparent that in the period $\epsilon t^{\frac{3}{2}(1+\nu)} = O(1)$ the modulation in the frequency gives rise to a *unit-order* phase shift, though its effect on the frequency itself is negligible. Equation (2.8*a*) is in fact the correct leading approximation to the surface pressure perturbation for the incipient transition period where

$$\epsilon t^{\frac{3}{2}(1+\nu)} = O(1).$$

Its alternative form (2.8*b*), on the other hand, suggests obviously a breakdown of the incipient transition solution at $\epsilon t^{(1+\nu)} = O(1)$ where the oscillation frequency is no longer close to that of $\cos\left[2t^{\frac{1}{2}(1+\nu)}\right]$. It may be noted in passing that the second non-uniformity may also be inferred from a comparison of the thicknesses of the shock layer and the entropy wake in the Cheng & Kirsch (1969) analysis; the two thicknesses become equal at $\epsilon t^{(1+\nu)} = O(1)$.

It is essential to point out that, to the leading order, the solutions of perturbation quantities for the finite-time period can be matched *directly* to those for the principal transition period, as subsequent analysis will confirm, and that, to the leading order, the latter *contains* the incipient transition solutions.† For the sake of conserving space, the formal analysis for the incipient transition period will be deleted from §§3 to 6.

2.4. *Two irreducible time scales*

Equations (2.8*a*) and (2.8*b*) suggest that the oscillation period in both transitions belongs to the same *order* as that of $\cos[2t^{\frac{1}{2}(1+\nu)}]$, i.e. $O[t^{\frac{1}{2}(1-\nu)}]$, which is *much shorter* than the incipient transition period by a factor of $O(\epsilon^{\frac{1}{3}})$, and than the principal transition period by a factor of $O(\epsilon^{\frac{1}{2}})$. Thus, associated with *each* transition period, there will be *two* irreducible, (asymptotically) distinct time scales. A proper description of the solution must therefore admit both time scales simultaneously. The suitable scale and the form of expansion for p may be inferred from (2.7) for either period (under suitable time scales noted above), in as much as a direct matching of the principal transition and finite-time solutions is anticipated. Forms for Y and other variables may be inferred in a similar manner.

The dynamics of the system during a short interval comparable to the smaller time scale is, of course, dominated by a nearly harmonic oscillation; however, the weak feed-back built up over a long period comparable to the larger time scale is sufficient to alter completely the oscillatory pattern. These facts are reflected in the method adopted by this study, which employs two time variables as in Cole & Kevorkian (1963, pp. 113–120) and Cole (1968). The procedure permits one to solve first the problem with the slow time variable held frozen, and to remove subsequently the non-uniformity associated with the 'fast event' through exercising the slow variable.

3. Basic construction and the proper variables of the principal transition period

From the foregoing discussion, we anticipate that the shock movement in the principal transition period is described by two variables, t and a relatively slow one $\tilde{t} \equiv \epsilon^{1/(1+\nu)}t$. A time change in the flow variable may then be attributed to two sources, e.g. $dY/dt = \partial Y/\partial t + \epsilon^{1/(1+\nu)}\partial Y/\partial\tilde{t}$. However, the two partial derivatives may have magnitudes different from the order *unity*, and this fact would lead to incorrect ordering in the asymptotic analysis. A relevant example is given by $Y = \cos[\omega(\tilde{t})\,t^{\frac{1}{2}(1+\nu)}]$, which leads to

$$\partial Y/\partial t = O[\omega Y t^{\frac{1}{2}(\nu-1)}] = O[\epsilon^{(1-\nu)/2(1+\nu)}] \ll O(1),$$

and

$$\partial Y/\partial\tilde{t} = O[\omega'(\tilde{t})\,Y t^{\frac{1}{2}(1+\nu)}] = O(\epsilon^{-\frac{1}{2}}).$$

However, a suitable transformation of the fast variable suffices to keep the partial derivatives bounded, as demonstrated in works of Kuzmak (1959) and Cole (1968).‡ Although the required transformation for the fast variable may be

† A detailed analysis of the incipient transition period, similar to that enunciated in §§3 to 6 has been carried out.

‡ The transformation $ds/dt = f(\tilde{t})$ in Kuzmak and Cole's analyses can not be directly applied to the present problem.

determined in the course of the analysis, it is expedient, in view of the foregoing example and (2.7), to assume a form $s = \omega(\tilde{t})\, t^{\frac{1}{2}(1+\nu)}$ for the new *swift* variable.

The solution structure must take into account the different entropy levels which exist in the field. Therefore, in analysing the transition period, we shall retain the two flow regions in Cheng & Kirsch's (1969) analysis so as to identify particles which have been involved in the earlier period, and introduce, in addition, an outer shock layer to accommodate newcomers. Hence the basic construction of the principal transition solution consists of three decks corresponding to three ranges of the Lagrangian variable, $Y_* \equiv y_*/b = O(\epsilon^{-1/(1+\nu)})$, $O(1)$, and $\exp[-|O(1/\epsilon)|]$, as shown in table 1.

The form of Lagrangian variables used for the intermediate and inner regions, Y_* and $\zeta = (Y_*^{1+\nu}/\sigma)^{2\epsilon}$, with $\sigma = 2 + (4\ln 2)\epsilon + \ldots$, are the same used by Cheng & Kirsch (1969) for their shock layer and entropy wake. The last column of the table indicates the relative physical thickness of the three regions, to be confirmed in §4.3.

	Fast variable	Slow variable	Lagrangian variable	Relative thickness in y
Outer (shock layer)	$s \equiv \omega(\tilde{t})t^{\frac{1}{2}(1+\nu)}$	$\tilde{t} \equiv \epsilon^{1/(1+\nu)}t$	$\eta \equiv \epsilon^{1/(1+\nu)}Y_*$	$O(1)$
Intermediate (inner shock layer)	s	$\tilde{t}$	Y_*	$O[\epsilon^{1/(1+\nu)}]$
Inner (entropy wake)	s	$\tilde{t}$	$\zeta \equiv (Y_*^{1+\nu}/\sigma)^{2\epsilon}$	$O(1)$

TABLE 1. The basic construction and the independent variables of the principal transition period

We remark in passing that the multiple-scale method could be applied to the present problem for a range $O(1) \leqslant t \leqslant O[\epsilon^{-1/(1+\nu)}]$ which is wider than the period $t = O[\epsilon^{-1/(1+\nu)}]$ considered. However, inclusion of the finite-time period would render the analysis unnecessarily cumbersome. In the subsequent analyses, which exclude the range $t = O(1)$, the initial condition is therefore replaced by the matching with the large-time limit of the finite-time solution corresponding to (2.7).

4. Structure of the principal transition period for $y_c \propto t$, $\nu = 0$

In this section and §5, the large-time planar piston problem, i.e. $y_c \propto t$, $\nu = 0$, is analysed.

4.1. *The outer region*

Substitution of $t = \tilde{t}/\epsilon^{1/(1+\nu)}$ into (2.7) and corresponding equations for other flow variables, with $\nu = 0$, leads to the following form of perturbation solutions for the outer region

$$\left.\begin{aligned} &\frac{y_{sh}}{b} - t = 1 + \tilde{t} + \epsilon^{\frac{2}{3}}\overline{Y}, \quad \frac{p\tau^2}{\rho_\infty b^2} = 1 + \epsilon^{\frac{2}{3}}\overline{p}, \quad \epsilon\frac{\rho}{\rho_\infty} = 1 + \epsilon^{\frac{2}{3}}\overline{\rho}, \\ &\frac{y - y_{sh}}{b} = \eta - \tilde{t} + \epsilon^{\frac{2}{3}}\overline{y}, \quad \frac{\tau(v - \dot{y}_{sh})}{\epsilon b} = -1 + \epsilon^{\frac{1}{3}}\overline{v}. \end{aligned}\right\} \tag{4.1}$$

All perturbation quantities denoted by an overbar, and their derivatives, will be assumed to be of unit order in this period. With s and $\tilde{t}$ as the fast and the slow variables, the first and second time derivatives may be expressed, respectively, as

$$\left.\begin{aligned} \frac{\partial}{\partial t} &= \frac{1}{2}\left(\frac{\epsilon}{\tilde{t}}\right)^{\frac{1}{2}}\Omega\frac{\partial}{\partial s}+\epsilon\frac{\partial}{\partial \tilde{t}}, \\ \frac{\partial^2}{\partial t^2} &= \frac{\epsilon}{4\tilde{t}}\Omega^2\frac{\partial^2}{\partial s^2}+\epsilon^{\frac{3}{2}}\left[\frac{\Omega}{\tilde{t}^{\frac{1}{2}}}\frac{\partial^2}{\partial s\,\partial\tilde{t}}+\frac{1}{2}\frac{d}{d\tilde{t}}\left(\frac{\Omega}{\tilde{t}^{\frac{1}{2}}}\right)\frac{\partial}{\partial s}\right]+\epsilon^2\frac{\partial^2}{\partial\tilde{t}^2}, \end{aligned}\right\} \tag{4.2}$$

where Ω is another slowly-varying function related to ω as

$$\Omega = \omega+2\tilde{t}\,d\omega/d\tilde{t}. \tag{4.3}$$

Equations (4.1), (4.2) and the shock condition (2.2) suggest expansions in ascending powers of $\epsilon^{\frac{1}{4}}$; e.g.

$$\overline{p} = \overline{p}_0(s,\tilde{t};\eta)+\epsilon^{\frac{1}{4}}\overline{p}_1(s,\tilde{t};\eta)+\epsilon^{\frac{1}{2}}\overline{p}_2(s,\tilde{t};\eta)+\epsilon^{\frac{3}{4}}\overline{p}_3(s,\tilde{t},\eta)+\dots. \tag{4.4}$$

We shall seek uniformly-valid expansions of this form for an *unbounded* s, and for $\tilde{t}$ and η in their unit-order ranges.†

With the first of (4.1), the shock location $Y_* = Y$ reads

$$\eta = \tilde{t}+\epsilon(1+\tilde{t})+\epsilon^{\frac{7}{4}}\overline{Y}, \tag{4.5}$$

the required boundary condition at the shock may be reduced to one at $\eta = \tilde{t}$:

$$\left.\begin{aligned} &\overline{p}_0 = 0, \quad \overline{p}_1 = 1, \quad \overline{p}_2 = \Omega\tilde{t}^{-\frac{1}{2}}\overline{Y}_{0s}, \quad \overline{p}_3 = \Omega\tilde{t}^{-\frac{1}{2}}\overline{Y}_{1s}, \\ &\overline{\rho}_0 = \overline{\rho}_1+1 = \overline{\rho}_2 = \overline{\rho}_3 = 0, \quad \overline{y}_0 = \overline{y}_1 = \overline{y}_2 = \overline{y}_3 = \dots = 0. \end{aligned}\right\} \tag{4.6}$$

Applying (4.1)–(4.3) to the equations of motion (2.1), we have

$$\left.\begin{aligned} &[\overline{Y}_i+\overline{y}_i]_{ss}+4\chi\overline{p}_{i\eta} = 0 \quad (i = 0,1), \\ &[\overline{Y}_2+\overline{y}_2]_{ss}+4\chi\overline{p}_{2\eta} = [(\overline{Y}_0+\overline{y}_0)_s-4\chi(\overline{Y}_0+\overline{y}_0)_{s\chi}]\,\chi^{-\frac{1}{2}}d\chi/d\tilde{t}, \end{aligned}\right\} \tag{4.7a}$$

$$\overline{y}_{i\eta}+\overline{\rho}_i = [\overline{\rho}_i-\overline{p}_i]_s = 0 \quad (i = 0,1,2), \tag{4.7b}$$

where the subscripts s and η signify partial differentiations and χ is a new slowly-varying function introduced to replace $\tilde{t}$

$$(\chi)^{\frac{1}{2}} \equiv \tilde{t}^{\frac{1}{2}}/\Omega = \tilde{t}^{\frac{1}{2}}/[2\tilde{t}(d\omega/d\tilde{t})+\omega]. \tag{4.8}$$

Eliminating $(Y+y)$ from (4.7a), we arrive at a system of de-coupled, linear hyperbolic equations in the variables s and η for the perturbation pressure field

$$\left.\begin{aligned} &\overline{p}_{iss}-4\chi\overline{p}_{i\eta\eta} = 0 \quad (i = 0,1), \\ &\overline{p}_{2ss}-4\chi\overline{p}_{2\eta\eta} = [\overline{p}_{0s}-4\chi\overline{p}_{0s\chi}]\,(d\chi/d\tilde{t})\,\chi^{-\frac{1}{2}} \equiv f(s,\eta,\chi), \end{aligned}\right\} \tag{4.9a}$$

with the outer boundary condition at $\eta = \Omega^2\chi$

$$\overline{p}_0 = 0, \quad \overline{p}_1 = 1, \quad \overline{p}_2 = \overline{Y}_s^{(0)}\chi^{-\frac{1}{2}}. \tag{4.9b}$$

Hence, with the slow variable appearing as a parameter, the pressure field is basically one in the classical *acoustic* theory.‡ We note, in view of (4.7a), that

† Although the solutions in their final forms necessarily belong to a class of 'general asymptotic expansions' (Erd'elyi 1961, Kaplun 1967) usage of such notion is avoided in the solution procedure of the multiple-scale techniques.

‡ Terms appearing in (4.9a) can all be generated from an acoustic equation. This is not to say, however, that the reduced problem can be explicitly solved by standard techniques.

the particle accelerations in the outer region no longer follow closely that of the shock as they do in a standard (non-linear) Newtonian theory.

The formulation for the pressure field at this stage is incomplete because of the lack of an inner boundary condition. With the energy equation, (4.7*b*), and the shock boundary conditions, we may relate the density, the particle position and velocity to the pressure field through

$$\left.\begin{aligned} &\bar{\rho}_0 = \bar{p}_0, \quad \bar{\rho}_1 = \bar{p}_1 - 2, \quad \bar{\rho}_2 = \bar{p}_2 - [\bar{Y}_{0s}]_* \chi^{-\frac{1}{2}}, \\ &\bar{y}_i = -\int_{\Omega^2\chi} \bar{\rho}_i \, d\eta \quad (i = 0, 1, 2), \\ &\bar{v}_0 = \bar{y}_{0s} 2^{-1}\chi^{-\frac{1}{2}}, \quad \bar{v}_1 = \bar{y}_{1s} 2^{-1}\chi^{-\frac{1}{2}}, \quad \bar{v}_2 = 2^{-1}\chi^{-\frac{1}{2}}\bar{y}_{2s} + (d\chi/d\bar{t})\,\bar{y}_{0\chi}, \end{aligned}\right\} \tag{4.10}$$

where, as in y_*, the subscript $_*$ in the second-order density correction refers to the condition at the shock when the particle in question first emerges behind it, therefore $[\bar{Y}_{0s}]_*$ is a function of η only.

The formal solution to the acoustic equation of the leading order, which satisfies the shock condition, is

$$\bar{p}_0 = \bar{\rho}_0 = P(s - 2^{-1}\chi^{-\frac{1}{2}}\eta; \chi) - P(s + 2^{-1}\chi^{-\frac{1}{2}}\eta - \Omega^2\chi^{\frac{1}{2}}; \chi). \tag{4.11a}$$

Through (4.10), we may relate the particle ordinate and velocity to the function $P(\xi; \chi)$

$$\bar{y}_0 = 2\chi^{\frac{1}{2}} \int_{s-\frac{1}{2}\Omega^2\chi^{\frac{1}{2}}}^{s-\eta^2-1\chi^{-\frac{1}{2}}} P(\xi;\chi)\, d\xi + 2\chi^{\frac{1}{2}} \int_{s-\frac{1}{2}\Omega^2\chi^{\frac{1}{2}}}^{s+\eta^2-1\chi^{-\frac{1}{2}}-\Omega^2\chi^{\frac{1}{2}}} P(\xi;\chi)\, d\xi, \tag{4.11b}$$

$$\bar{v}_0 = P(s - 2^{-1}\chi^{-\frac{1}{2}}\eta; \chi) + P(s + 2^{-1}\chi^{-\frac{1}{2}}\eta - \Omega^2\chi^{\frac{1}{2}}; \chi) - 2P(s - \tfrac{1}{2}\Omega^2\chi^{\frac{1}{2}}; \chi). \tag{4.11c}$$

Using (4.11*a*) and (4.11*b*) in the equation of motion (4.7*a*), and integrating twice with respect to the fast variable, we have

$$\bar{Y}_0 = 4\chi^{\frac{1}{2}} \int_0^{s-\frac{1}{2}\Omega^2\chi^{\frac{1}{2}}} P(\xi;\chi)\, d\xi + A(\chi)\, s + B(\chi), \tag{4.11d}$$

where the constants of integration A and B are functions of χ. The determination of $P(\xi; \chi)$, A and B requires an inner boundary condition.

The forms of the first-order corrections, $\bar{p}_1$, etc., remain the same as (4.11), except that a numeral one must be added to the right of (4.11*a*) for the pressure, that the same be subtracted for the density, and that a term $(\eta - \Omega^2\chi)$ must be added to the right of (4.11*b*) for the particle co-ordinate. Although the second-order pressure correction is not of immediate interest, its uniformity is crucial to the determination of the leading-order solution, as previously noted. An analysis of the second-order problem is therefore carried out below.

It is expedient to write the equation governing $\bar{p}_2$ as

$$4 \frac{\partial^2}{\partial z\, \partial \bar{z}} \bar{p}_2 = f(s, \eta; \chi) \equiv F(z, \bar{z}; \chi), \tag{4.12}$$

where $z \equiv s - 2^{-1}\chi^{-\frac{1}{2}}\eta$; $\bar{z} \equiv s + 2^{-1}\chi^{-\frac{1}{2}}\eta$. A particular integral can be readily written which could however contain terms unbounded like s^2 and s. It turns out that the apparent divergence at this stage can be avoided by adding suitable homogeneous solutions. One may thus arrive at a solution satisfying both (4.12) and the shock condition (4.9*b*), which is bounded throughout the period $\tilde{t} = O(1)$:

$$\bar{p}_2 = P_2(z,\chi) - P_2(\bar{\bar{z}} - \Omega^2\chi^{\frac{1}{2}};\chi) + G(z,\bar{\bar{z}};\chi) - G(\bar{\bar{z}} - \Omega^2\chi^{\frac{1}{2}},\bar{\bar{z}};\chi) + 4P(\bar{\bar{z}} - \Omega^2\chi^{\frac{1}{2}};\chi) + A(\chi)\chi^{-\frac{1}{2}}, \quad (4.13)$$

where $P_2(z;\chi)$ designates a homogeneous solution for $\bar{p}_2$, and the last two terms result directly from the outer boundary condition. The function $G(z,\bar{\bar{z}};\chi)$ is the *particular* solution

$$\begin{aligned} G(z,\bar{\bar{z}};\chi) \equiv {} & \frac{\bar{\bar{z}}-z}{4\chi^{\frac{1}{2}}}\frac{d\chi}{d\tilde{t}}[P(z;\chi) + P(\bar{\bar{z}} - \Omega^2\chi^{\frac{1}{2}};\chi)] - \chi^{\frac{1}{2}}(\bar{\bar{z}}-z)\frac{d\chi}{d\tilde{t}} \\ & \times\Big\{P_\chi(z;\chi) + P_\chi(\bar{\bar{z}} - \Omega^2\chi^{\frac{1}{2}};\chi) - P'(\bar{\bar{z}} - \Omega^2\chi^{\frac{1}{2}};\chi)\frac{d}{d\chi}(\Omega^2\chi^{\frac{1}{2}}) + \frac{\bar{\bar{z}}-z}{8\chi} \\ & \times[P'(z,\chi) - P'(\bar{\bar{z}} - \Omega^2\chi^{\frac{1}{2}};\chi)] + \frac{1}{4\chi}[Pz(;\chi) + P(\bar{\bar{z}} - \Omega^2\chi^{\frac{1}{2}};\chi)]\Big\}, \end{aligned} \quad (4.14)$$

where the prime on P signifies the partial derivative with respect to the first argument, and the subscript χ, the partial derivative with respect to the second argument of P. This form of particular solution is chosen so that G vanishes at the shock and remains finite for unbounded s, as long as $\eta, \tilde{t}, P, P'$ and P_χ remain so.

4.2. *The intermediate and inner regions*

In order to match with the solution of the outer region, and with that of the shock layer in the earlier period, it is necessary to assume for the intermediate region [$Y_* = O(1)$, refer to table 1] perturbations of the following form

$$\left.\begin{aligned} p\tau^2/\rho_\infty b^2 &= 1 + \epsilon^{\frac{3}{2}}\bar{\bar{p}}, \epsilon\rho/\rho_\infty = \bar{\bar{\rho}}, \\ (y - y_{sh})/b &= -\tilde{t} + \epsilon^{\frac{3}{2}}h(s,\tilde{t}) + \epsilon\bar{\bar{y}}, \\ (v - \dot{y}_{sh})\tau/\epsilon b &= -1 + \epsilon^{\frac{1}{2}}\bar{\bar{v}}, \end{aligned}\right\} \quad (4.15)$$

where the double overbarred quantities will be taken to be of unit order. We again seek expansions in ascending powers of $\epsilon^{\frac{1}{2}}$. Noting that Y_* in this region is an order ϵ higher than that in the outer region, it follows from (2.1) that $\partial\bar{\bar{p}}/\partial Y_* = O(\epsilon)$; hence, matching with the outer pressure gives $\bar{\bar{p}} = \bar{p}(s,0;\chi;\epsilon)$ up to the third-order correction. The density is determined from the particle isentropic relation $\bar{\bar{\rho}} = \dot{Y}_*^{-2}[1 + O(\epsilon^{\frac{3}{2}})]$, where $\dot{Y}_*$ is dY/dt at $Y = Y_* = O(1)$. The particle ordinate may then be integrated as $\bar{\bar{y}} = \int \dot{Y}_*^2\,dY_* + g(s,\chi;\epsilon) + O(\epsilon^{\frac{3}{2}})$. Matching the outer limit of $\bar{\bar{y}}$ with the inner limit of $\bar{y}$ of §4.2 determines $h(s,\chi)$ and $g(s,\chi;\epsilon)$ in terms of the expansions of $\bar{y}$, so that we can write

$$\begin{aligned} y/b = {} & t + 1 + \epsilon^{\frac{3}{2}}[\bar{Y}_0 + \bar{y}_0]_{\eta\to 0} + \epsilon[\bar{Y}_1 + \bar{y}_1]_{\eta\to 0} + \epsilon^{\frac{5}{4}}[\bar{Y}_2 + \bar{y}_2]_{\eta\to 0} \\ & - \epsilon\int^\infty (\dot{Y}_*^2 - 1)\,dY_* + O(\epsilon^{\frac{3}{2}}), \end{aligned} \quad (4.16)$$

where the last term is an indefinite integral. The existence of the $\bar{y}$ expansions in the limit $\eta \to 0$ can be verified to the second-order corrections on the basis of the result from §4.1. Note that $\dot{Y}_*^2$, hence the integral, depends only on the shock motion in the earlier (finite-time) period. According to (2.7),

$$|\dot{Y}_*^2 - 1| = O(Y_*^{-\frac{1}{4}})$$

as $Y_* \to \infty$, therefore, the integral in (4.16) exists in the upper limit. In the inner limit, $Y_* \to 0$, $\dot{Y}_*^2$ becomes infinite like $2Y_*^{-1}$ (Cheng & Kirsch 1969), and the integral of (4.16) becomes

$$\int^{\infty} (\dot{Y}_*^2 - 1)\, dY_* \sim -2 \ln Y_* + \text{F.P.} \int_0^{\infty} \dot{Y}_*^2 \, dY_*. \tag{4.17}$$

The last integral of (4.17), with F.P., the finite part, applied to both limits of the integral, defines a thickness of the intermediate layer of which the mean location may be identified with the first line on the right of (4.16).

In view of the independent variables s, $\bar{t}$ and ζ of table 1, and the identification of the inner region with the entropy wake, we anticipate the perturbation quantities in the inner region to take the following forms:

$$\left.\begin{aligned} p\tau^2/\rho_\infty b^2 = 1 + \epsilon^{\frac{3}{4}}\tilde{p}, \quad \epsilon\rho/\rho_\infty \zeta^{(1/2\epsilon)-1} = \tilde{\rho}, \\ y/b - t = \tilde{y}, \quad \tau v/b = 1 + \epsilon^{\frac{1}{2}}\tilde{v}, \end{aligned}\right\} \tag{4.18}$$

with all tilde quantities taken to be of unit order. It follows from the equations of motion that the $\tilde{p}$ expansion can be directly matched to the expansion of $\bar{\bar{p}}$, hence $\bar{p}$ in the limit $\eta \to 0$. The particle-isentropic condition then yields

$$\tilde{\rho} = [1 + \epsilon^{\frac{3}{4}}\tilde{p}]^{1-2\epsilon}/[1 + O(\epsilon^2)],$$

which gives the particle displacement

$$y/b = t + \zeta\{1 + (2\ln 2)\,\epsilon - \epsilon^{\frac{3}{4}}[\bar{p}_0 + \epsilon^{\frac{1}{4}}\bar{p}_1 + \epsilon^{\frac{1}{2}}\bar{p}_2 + \ldots]_{\eta \to 0}\}, \tag{4.19}$$

where the inner boundary condition $\tilde{y} = 0$ at $\zeta = 0$ has also been satisfied.

Comparing (4.19) in the limit $\zeta \to 1$ with (4.16) in the limit $Y_* \to 0$ shows that matching of the two inner regions in y can be established provided

$$\left.\begin{aligned} &[\bar{p}_0 + \bar{\bar{Y}}_0 + \bar{y}_0]_{\eta\to 0} = 0, \quad [\bar{p}_1 + \bar{\bar{Y}}_1 + \bar{y}_1]_{\eta \to 0} = \text{F.P.} \int_0^{\infty} \dot{Y}_*^2 \, dY_*, \\ &[\bar{p}_2 + \bar{\bar{Y}}_2 + \bar{y}_2]_{\eta\to 0} = 0, \quad [\bar{p}_3 + \bar{\bar{Y}}_3 + \text{F.P.}\, \bar{y}_3]_{\eta \to 0} = [p_0^2]_{\eta \to 0}, \ldots \end{aligned}\right\} \tag{4.20}$$

Equations (4.20), which involve only the inner limit $\eta \to 0$ of the outer solution, are therefore the inner boundary conditions for the P's desired (for subsequent discussion, the result for the third-order correction is also included). With (4.20), matching of the two inner regions in other flow variables is readily verified.

Adopting Cheng & Kirsch's (1969) definition of the outer-edge ordinate for the entropy wake, Y_e, the four equations shown in (4.20) can be combined to yield

$$(\bar{p} - 1) + \tfrac{1}{2}(\bar{p} - 1)^2 + (Y_e - t - 1) + \tfrac{1}{2}(Y_e - t - 1)^2 = O(\epsilon^2), \tag{4.21}$$

which, in the transition period, agrees with the *pressure-volume relation* derived previously for the explosion-controlled régime. From a mechanics viewpoint, (4.20) is interesting in that the inner region, if being looked upon as an elastic body, responds to the external pressure change according to Hooke's law.

5. Final analysis of the principal transition period

5.1. *Reduction to an integral-difference equation*

Substitution of (4.11*a*), (4.11*b*) and (4.11*d*) into the first of the pressure-volume relations (4.20) give a linear integral-difference equation

$$P(s;\chi)-P(s-\Omega^2\chi^{\frac{1}{2}};\chi)+2\chi^{\frac{1}{2}}\int_0^s P(\xi;\chi)\,d\xi$$
$$+2\chi^{\frac{1}{2}}\int_0^{s-\Omega^2\chi^{\frac{1}{2}}} P(\xi;\chi)\,d\xi+A(\chi)\,s+B(\chi)=0, \qquad (5.1)$$

which implies a differential-difference equation (D.D.E.) for $P(\xi;\chi)$

$$P''(s;\chi)-P''(s-\Omega^2\chi^{\frac{1}{2}};\chi)+2\chi^{\frac{1}{2}}P'(s;\chi)+2\chi^{\frac{1}{2}}P'(s-\Omega^2\chi^{\frac{1}{2}};\chi)=0, \qquad (5.1a)$$

where, as before, the prime signifies a differentiation with respect to the first argument.

Equation (5.1) admits a solution sinusoidal in s, for an unbounded s, of the form†

$$P(s;\chi)=C(\chi)\,e^{i\alpha s}+D(\chi), \qquad (5.2)$$

provided that

$$2\chi^{\frac{1}{2}}/\alpha=\tan\left[\tfrac{1}{2}\alpha\Omega^2\chi^{\frac{1}{2}}\right] \qquad (5.3)$$

and that

$$A=-4\chi^{\frac{1}{2}}D, \quad B=2\chi^{\frac{1}{2}}[\Omega^2\chi^{\frac{1}{2}}D-i(2C/\alpha)]. \qquad (5.4)$$

We observe that, in order to match with the oscillatory behaviour (2.7), α in (5.2)–(5.4) must be real. More important is the fact that α must be independent of χ, otherwise partial χ derivatives of P would be unbounded. Since $\Omega^2\chi=\tilde{t}$, the requirement (5.3) may be alternatively written as

$$\tilde{t}=\mu[\tan^{-1}\mu-n\pi], \quad \mu\equiv 2\chi^{\frac{1}{2}}/|\alpha|, \qquad (5.5)$$

where $\tan^{-1}\mu$ refers to the principal value of the arc tangent, and n is an integer including zero. For reason to be given below, only the principal branch of the solution ($n=0$) is used. Since $\Omega\equiv\omega+2\tilde{t}\,d\omega/d\tilde{t}$, (5.3) or (5.5) yields a differential equation for the slow function $\omega(\tilde{t})$ needed for the transformation $t\rightarrow s$. This gives

$$\omega_1(\tilde{t})\equiv|\alpha|\omega(\tilde{t})=\frac{1}{\tilde{t}^{\frac{1}{2}}}\int_0\frac{d\tilde{t}}{\mu(\tilde{t})}=\frac{1}{\tilde{t}^{\frac{1}{2}}}\left\{\tan^{-1}\mu+\int_0\frac{\tan^{-1}\mu\,d\mu}{\mu}\right\}, \qquad (5.6)$$

which is independent of α and completely defines $\alpha s=\omega_1(\tilde{t})\,t^{\frac{1}{2}}$ in (5.2). For small $\tilde{t}$, $\mu\sim\tilde{t}^{\frac{1}{2}}$, (5.6) then yields $|\alpha|\omega\sim 2$, thus recovering the dominant oscillatory behaviour of (2.7). It must be pointed out that, omitted from (5.6) is a constant of integration, inclusion of which would have resulted in a phase shift belonging to the order $\epsilon^{-\frac{1}{2}}$ which has no counterpart in (2.7). Not considered in (5.6) are the branch values of $\mu(\tilde{t})$ other than $n=0$, which would, however, give rise to an unwanted behaviour $P\sim C\exp[\pm i2t^{\frac{1}{2}}\ln\tilde{t}]+D$ for small $\tilde{t}$.

† We note that if $s=O(1)$, a complete solution to (5.1) would require a knowledge of the initial data specified over the interval $-\Omega^2\chi^{\frac{1}{2}}\leqslant s\leqslant 0$, and that even for an unbounded s, a formal proof of the uniqueness of a bounded solution is not available (see Bellman & Cooke 1963). The form of (5.2) assures, nevertheless, that P and all its partial derivatives, are finite for an unbounded s.

The slow functions $C(\chi)$ and $D(\chi)$ remain undetermined. An examination of (4.11) reveals, however, that the slow function $D(\chi)$ of (5.2) can be left arbitrary, since it never appears in solutions representing physical quantities.

The corresponding results for the next order are of the same forms, except that the terms

$$\left[\Omega^2\chi^{\frac{1}{2}} - 1 - \text{F.P.}\int_0^\infty \dot{Y}_*^2\, dY_*\right]$$

have to be added to the right of (5.1) and the second of (5.4).

5.2. *Secular terms and final solutions*

To expedite the determination of $C(\chi)$, we may differentiate the second and third of the pressure–volume relation (4.20) twice with respect to s, so that

$$\overline{p}_{1ss} + (\overline{Y}_1 + \overline{y}_1)_{ss} = \overline{p}_{2ss} + \overline{Y}_{2ss} + \overline{y}_{2ss} = 0 \quad (\eta \to 0).$$

The acceleration terms may be eliminated through the equations of motion (4.7a), and the first of (4.20). This results in the *inner* boundary conditions at $\eta \to 0$ involving the $\overline{p}$'s alone

$$\overline{p}_{1ss} - 4\chi\overline{p}_{1\eta} = \overline{p}_{2ss} - 4\chi\overline{p}_{2\eta} - (\overline{p}_{0s} - 4\chi\overline{p}_{0s\chi})\,\chi^{-\frac{1}{2}}(d\chi/d\tilde{t}) = 0, \tag{5.7}$$

which, through (4.9a), may be written alternatively as

$$\overline{p}_{1\eta\eta} - \overline{p}_{1\eta} = \overline{p}_{2\eta\eta} - p_{2\eta} = 0, \quad (\eta \to 0). \tag{5.7a}$$

Applying (5.7) to the results of §4.2, we arrive at equations comparable to the D.D.E. (5.1a)

$$\left.\begin{aligned} P_1'' - P_{1r}'' + 2\chi^{\frac{1}{2}}(P_1' + P_{1r}') &= 0,\\ P_2'' - P_{2r}'' + 2\chi^{\frac{1}{2}}(P_2' + P_{2r}') &= W(s;\chi),\end{aligned}\right\} \tag{5.8}$$

where the prime, as before, stands for differentiation with respect to s, and the subscript r signifies a *retarded* s, e.g. $P_r = P(s - \Omega^2\chi^{\frac{1}{2}};\chi)$. The forcing function $W(s;\chi)$ depends on the leading-order solution P,† i.e. on $P = C(\chi)\,e^{i\alpha s} + D(\chi)$, and may be expressed as

$$W = -\frac{1}{\chi^{\frac{1}{2}}}\frac{d\chi}{d\tilde{t}}\left\{8\chi^{\frac{3}{2}}\frac{dD}{d\chi} - e^{i\alpha s}\left[f(\chi)\frac{dC}{d\chi} + J(\chi)C\right]\right\}, \tag{5.9}$$

with

$$f(\chi) \equiv -2\chi(i\alpha)\,(1 - e^{-i\alpha\Omega^2\chi^{\frac{1}{2}}}) + 2\chi^{\frac{3}{2}}\Omega^2\,[2\chi^{\frac{1}{2}}(i\alpha) + \alpha^2]\,e^{-i\alpha\Omega^2\chi^{\frac{1}{2}}},$$

$$\begin{aligned} J(\chi) \equiv{}& (i\alpha)(1 - e^{-i\alpha\Omega^2\chi^{\frac{1}{2}}}) + \frac{d(\Omega^2\chi^{\frac{1}{2}})}{d\chi}\,[2\chi\alpha^2(2 + \Omega^2\chi) + (i\alpha)\,\chi^{\frac{3}{2}}(4 - \alpha^2\Omega^2)]e^{-i\alpha\Omega^2\chi^{\frac{1}{2}}}\\ &+ 4\chi^{\frac{1}{2}}\frac{d}{d\chi}(\chi\Omega^2)[\alpha^2 + 2\chi^{\frac{1}{2}}(i\alpha)]\,e^{-i\alpha\Omega^2\chi^{\frac{1}{2}}}.\end{aligned}$$

† $$\begin{aligned} W \equiv{}& \chi^{-\frac{1}{2}}\frac{d\chi}{dt}\cdot\Big\{2\Omega^2\chi^{\frac{3}{2}}(2\chi^{\frac{1}{2}}P'_{r\chi} - P''_{r\chi}) - 4\chi^{\frac{3}{2}}(P_\chi + P_{r\chi}) - 4\chi(P'_\chi - P'_{r\chi}) + (P' - P'_r)\\ &+ \left[\frac{d}{d\chi}(\Omega^2\chi^{\frac{1}{2}})\right]\cdot[4\chi(\chi^{\frac{1}{2}}P'_r - P''_r) + \Omega^2\chi^{\frac{3}{2}}(P'''_r - 2\chi^{\frac{1}{2}}P''_r)]\Big\} + 4(2\chi^{\frac{1}{2}}P'_r - P''_r),\end{aligned}$$

$P_\chi \equiv \partial P(s;\chi)/\partial\chi, \quad P_{r\chi} \equiv [\partial P(\xi;\chi)/\partial\chi]_{\xi\to(s-\Omega^2\chi^{\frac{1}{2}})}.$

Note: $\alpha\Omega^2\chi^{\frac{1}{2}} = 2\tan^{-1}(2\chi^{\frac{1}{2}}/\alpha)$; $d(\Omega^2\chi^{\frac{1}{2}})/d\chi = 2\chi^{-\frac{1}{2}}(\alpha^2 + 4\chi)^{-1}$.

Hence the forcing function in the D.D.E. consists of two terms, with one being independent of s, and the other being proportional to $e^{i\alpha s}$. The first term gives a particular solution proportional to s, and the second leads to resonant solution proportional to $s\,e^{i\alpha s}$. Therefore, the second-order correction would be unbounded with s, unless W vanishes identically. The last provision requires

$$dD/d\chi = 0, \quad f(\chi)\, dC/d\chi + J(\chi)\, C = 0. \tag{5.10}$$

Integrating (5.10) with respect to the slow variable χ, which except in (5.6) has remained dormant up to this stage (!), gives the required form of $C(\chi)$ and $D(\chi)$ for the uniformity of the expansion (3.4) with respect to the unbounded s

$$D = D_0,\ \text{const.}, \quad C(\chi) = \exp\left\{-\int_a (J/f)\, d\chi\right\}, \tag{5.11}$$

where D_0 and a are constants of integration, to be determined presently.

Inspection of (5.3), (5.6) and (5.10) shows that $\alpha\omega$, χ/α^2, $\Omega^2\alpha^2$, f/α^3, J/α, C and D, hence P, are invariant with respect to α except for a dependence on its sign. It follows that, in terms of t and $\tilde{t}$, $C(\chi)$ has only two forms, say, $C^+(\tilde{t})$ and $C^-(\tilde{t})$. The entire family of solutions for $-\infty < \alpha < \infty$ are represented by

$$P = P^+(t;\tilde{t}) + P^-(t;\tilde{t}) = C^+(\tilde{t})\, e^{i\omega_1(\tilde{t})t^{\frac12}} + C^-(\tilde{t})\, e^{-i\omega_1(\tilde{t})t^{\frac12}} + D_0. \tag{5.12}$$

In the inner limit $\tilde{t} \to 0$, with $\chi \sim \frac14 \alpha^2 \tilde{t}$, $\Omega \sim 2/\alpha$, $f \sim 16\chi^{\frac32} + O(\chi^2)$, and $J \sim 20\chi^{\frac12} + O(\chi)$, (5.11) gives

$$\left.\begin{aligned} C^{\pm}(\tilde{t}) &\sim \frac{C_0^{\pm}}{\tilde{t}^{\frac54}}\,[1 + O(\tilde{t}^{\frac12})], \\ \alpha\omega &= \pm\,\omega_1(\tilde{t}) \sim \pm\, 2[1 - \tilde{t}/18], \end{aligned}\right\} \tag{5.13}$$

where C_0^+ and C_0^- are the constants of integration (replacing a) for C^+ and C^-, respectively. With (5.13), (4.11 a) yields the perturbation pressure in the limit $\tilde{t} \to 0$

$$\overline{p} \sim \frac{2}{\tilde{t}^{\frac34}}\{C_0^+ \exp[i(2t^{\frac12} + \tfrac12\pi)] + C_0^- \exp[-i(2t^{\frac12} + \tfrac12\pi)]\}\,\{1 - (\eta/\tilde{t})\}\left\{1 - \frac{i}{18}[\tilde{t}^3/\epsilon]^{\frac12} + \ldots\right\}. \tag{5.14}$$

This is to be compared with the large-time limit ($t \to \infty$) of the finite-time solution (Cheng & Kirsch 1969) corresponding to (2.7), written in terms of t, $\tilde{t}$ and $\overline{p}$,

$$\overline{p} \sim -\{(A_0/\tilde{t}^{\frac34}) \cos(2t^{\frac12} + \phi_0) + B_0[\tilde{t}^3/\epsilon]^{\frac14} \sin(2t^{\frac12} + \phi_0) + \ldots\}\,[1 - (\eta/\tilde{t})]. \tag{5.15}$$

A common domain of validity exists for (5.14) and (5.15) in the range $\epsilon \ll \tilde{t} \ll \epsilon^{\frac13}$ (for $0 < \eta \leqslant \tilde{t}$ and an unbounded s) where matching identifies

$$C_0^+ = \tfrac14 A_0\, e^{i(\phi_0 + \frac12\pi)}, \quad C_0^- = \tfrac14 A_0\, e^{-i(\phi_0 + \frac12\pi)}. \tag{5.16}$$

The function $P(s;\chi)$ is now completely determined and can be written as

$$P(s;\chi) = (A_0/\mu^{\frac52})\, e^{-\kappa(\mu)} \sin(|\alpha| s - \sigma + \phi_0) \tag{5.17}$$

where, as before, μ is the inverse of $\tilde{t} = \mu \tan^{-1}\mu$, and

$$\kappa(\mu) \equiv \mathrm{Re}\int_0 \left[\frac{J}{f} - \frac{5}{4\chi}\right] d\chi = \int_0 \left\{\frac{d\mu}{(1+\mu^2)\,[(1+\mu^2)\tan^{-1}\mu + \mu]} - \frac12 \frac{d\mu}{\mu}\right\},$$

$$\sigma(\mu) \equiv \mathrm{Im}\int_0 \left[\frac{J}{f} - \frac{5}{4\chi}\right] d\chi = -\tan^{-1}\mu.$$

With (5.17) and (4.11a), we arrive at the complete pressure field p in the transition period, written as a function of t and $\tilde{t}$,

$$\overline{p}_0 = -A_0(e^{-\kappa(\mu)}/\mu^{\frac{5}{2}})\cos\{\omega_1(\tilde{t})\,t^{\frac{1}{2}}+\phi_0\}\sin\{\tan^{-1}\mu[1-(\eta/\tilde{t})]\}, \qquad (5.18)$$

where $\omega_1(\tilde{t})$ has been defined in (5.6). Corresponding solutions for other flow quantities may be obtained through (5.17) and (4.11).

5.3. *Small and large time behaviour*

The foregoing analysis gives a valid description (under the Newtonian approximation) of the transition from an explosion-controlled régime to the final reattachment of the shock layer. Because of the two-time character of the solution in $\omega_1(\tilde{t})\,t^{\frac{1}{2}}$, application of (5.18) to the physical problem (with ϵt substituting $\tilde{t}$) reveals that the range of $\tilde{t} \ll 1$, as well as that of $\tilde{t} \gg 1$, contain in themselves several rather distinct stages. This observation should have been apparent from the form of $\overline{p}$ given in (5.14) for the inner limit $\tilde{t} \to 0$; namely, the behaviour

$$\overline{p} \sim -(A_0/\tilde{t}^{\frac{3}{4}})\cos(2t^{\frac{1}{2}}+\phi_0)\,[1-(\eta/\tilde{t})], \qquad (5.19)$$

does not hold for all $\tilde{t} \ll 1$, its validity, according to (5.14) requires $\tilde{t} \ll \epsilon^{\frac{1}{3}}$. For $t = O(\epsilon^{\frac{1}{3}})$, noting that $\omega_1 \sim 2-\tilde{t}/9$, the argument under the cosine in (5.18) becomes $[2t^{\frac{1}{2}}+\phi_0-\tilde{t}^{\frac{3}{2}}/9\epsilon^{\frac{1}{2}}]$, where the third term inside the bracket contributes to a unit-order phase shift and therefore cannot be omitted. It is apparent that the departure from the explosion-controlled régime at small t must involve at least three distinct stages: $\epsilon \ll \tilde{t} \ll \epsilon^{\frac{1}{3}}$, $\tilde{t} = O(\epsilon^{\frac{1}{3}})$ and $\epsilon^{\frac{1}{3}} \ll \tilde{t} \ll 1$. The departure in $\overline{p}$ is small in the first stage, is not small (but mainly in the form of a phase shift) in the second stage, and, in the third stage, both frequency and damping rate vary.

The stage at $\tilde{t} = O(\epsilon^{-\frac{1}{3}})$, i.e. $t = O(\epsilon^{\frac{2}{3}})$, coincides with the *incipient transition period* mentioned in §§2.3 and 2.4. It is important to point out that an asymptotic analysis of this period in the leading order, based on a slow-time scale of the order $\epsilon^{-\frac{2}{3}}$, recovers precisely (2.8a) and the phase shift noted above. This confirms the statement made earlier that the incipient transition solution is contained in the principal transition solution (at least for the leading approximations for p, y, etc.).

For $\tilde{t} \gg 1$, one has $\mu \sim 2\tilde{t}/\pi$ and

$$\left.\begin{aligned} \kappa &\sim -\tfrac{1}{2}\ln\mu + \text{F.P.}\,\kappa(\infty) + O(\mu^{-3}), \\ \tilde{t}^{\frac{1}{2}}\omega_1 &\sim \tfrac{1}{2}\pi\ln\mu + \tfrac{1}{2}\pi + O(\mu^{-3}). \end{aligned}\right\} \qquad (5.20)$$

Owing to the two-time character of the solution, the transition at large $\tilde{t}$ again gives rise to a number of distinct stages. The last stage of transition to the limit $\tilde{t} \to \infty$, according to (5.18) is

$$\overline{p}_0 \sim -(\tfrac{1}{2}\pi)^2 A_0\, e^{-\text{F.P.}\,\kappa(\infty)}\,\frac{1}{\tilde{t}^2}\cos\left\{\frac{\pi}{2\epsilon^{\frac{1}{2}}}\ln\left(\frac{2\tilde{t}}{\pi}\right)+\frac{\pi}{2\epsilon^{\frac{1}{2}}}+\phi_0\right\}\cos\{\pi\eta/2\tilde{t}\}, \qquad (5.21)$$

which, to be sure, requires $\epsilon^{\frac{1}{2}}\mu^3 \gg 1$, i.e. $\epsilon^{\frac{1}{2}}\tilde{t}^3 \gg 1$. One sees from above that, even towards the end of the transition, the oscillation persists and the damping rate remains algebraic. But both frequency and amplitude are reduced at rates much faster than those for small $\tilde{t}$ [compare (5.21) with (5.19)].

6. The problem of an expanding cylinder ($y_c \propto t$, $\nu = 1$)

In this section, the corresponding cylindrical problem will be presented. Much of the detail will be forsaken to conserve space; only major differences from the plane case will be emphasized.

From the large-time behaviour of the finite-time solution, e.g. (2.7), and the form of independent variables given in table 1 in §3 the following expansions are assumed for the outer region.

$$\left.\begin{aligned} (y_{sh}/b)-t &= 1/(2t)+\epsilon(t/2)+\epsilon\overline{Y}_0+\epsilon^{\frac{3}{2}}\overline{Y}_1+\dots,\\ (p\tau^2)/(\rho_\infty b^2) &= 1+\epsilon^{\frac{1}{2}}\overline{p}_0+\epsilon\overline{p}_1+\dots,\\ (y-y_{sh})/b &= \epsilon^{\frac{1}{2}}(\eta^2-\tilde{t}^2)/(2\tilde{t})+\epsilon\overline{y}_0+\epsilon^{\frac{3}{2}}\overline{y}_1+\dots. \end{aligned}\right\} \tag{6.1}$$

It is noted that the perturbations progress in $\frac{1}{2}$ powers of ϵ, instead of $\frac{1}{4}$ powers as in the plane case. The expansions for ρ and v can be easily inferred.

With the expansions (6.1), the exact problem (2.1) can be manipulated to yield the equations for the pressure perturbations

$$\left.\begin{aligned} &\frac{\partial^2\overline{p}_0}{\partial s^2}-4\chi_1^2\frac{\partial^2\overline{p}_0}{\partial\eta_1^2}=0,\\ &\frac{\partial^2\overline{p}_1}{\partial s^2}-4\chi_1^2\frac{\partial^2\overline{p}_1}{\partial\eta_1^2}=\frac{\partial^2}{\partial s^2}[\overline{p}_0]^2-\frac{2}{\Omega_1}\frac{\partial^2}{\partial s\partial\tilde{t}}\overline{p}_0+\left[\frac{2}{\Omega_1\tilde{t}}-\frac{1}{\Omega_1^2}\frac{d\Omega_1}{d\tilde{t}}\right]\frac{\partial\overline{p}_0}{\partial s}+\frac{2}{\Omega_1^2\tilde{t}^2}, \end{aligned}\right\} \tag{6.2}$$

where $\eta_1 \equiv \eta^2$, $\chi_1 \equiv \tilde{t}/\Omega_1 \equiv \tilde{t}/[\tilde{t}(d\omega/d\tilde{t})+\omega]$. These equations are analogous to (4.9a), except that the perturbation $\overline{p}_1$ in the cylindrical case is influenced by the non-linear effect $(\overline{p}_0)^2$. In terms of the characteristic variables $z_1 = s-\eta_1/(2\chi_1)$ and $\bar{\bar{z}}_1 = s+\eta_1/(2\chi_1)$, the solution of $\overline{p}_0$ can be readily given in terms of a known function Q

$$\overline{p}_0 = Q(z_1,\tilde{t})-Q(z_1-\tilde{t}^2/\chi_1,\tilde{t}), \tag{6.3}$$

which satisfies the first-order shock condition $\overline{p}_0 = 0$ at $\eta_1 = \tilde{t}^2$. Corresponding results for the streamline displacement and shock position are

$$\left.\begin{aligned} \overline{y}_0 &= \frac{1}{\Omega_1}\left\{\int_{s-\frac{1}{2}(\Omega_1\tilde{t})}^{s-\eta_1/(2\chi_1)}Q(\xi,t)\,d\xi+\int_{s-\frac{1}{2}(\Omega_1\tilde{t})}^{s+\eta_1/(2\chi_1)-\tilde{t}^2/\chi_1}Q(\xi,t)\,d\xi\right\},\\ \overline{Y}_0 &= \frac{2}{\Omega_1}\int_0^{s-\frac{1}{2}(\Omega_1\tilde{t})}Q(\xi,t)\,d\xi+A_1(\tilde{t})\,s+B_1(\tilde{t}). \end{aligned}\right\} \tag{6.4}$$

Finally, $\overline{p}_1$ can be obtained for (6.2) in the form

$$\begin{aligned} \overline{p}_1 = Q_1(z_1,\chi_1)-Q_1(\bar{\bar{z}}_1-\Omega_1^2\chi_1,\chi_1)+H(z_1,\bar{\bar{z}}_1,\chi_1)-H(\bar{\bar{z}}_1-\Omega_1^2\chi_1,z_1,\chi_1)\\ +4Q(\bar{\bar{z}}_1-\Omega_1^2\chi_1,\chi_1)+2\Omega_1A_1(\tilde{t})-1/(\Omega_1\chi_1)^2, \end{aligned} \tag{6.5}$$

where the undetermined function Q_1 is the complementary solution, and the last three terms are due to the non-homogeneous first-order correction in the shock condition. The function H is a particular solution. We shall forsake the details of H, except to mention that its form, like the G of (4.14), is chosen so that H vanishes at $\eta_1 = \tilde{t}^2$ and is finite for unbounded s.

In the intermediate region, y_* is of order unity and it can be shown from (2.1) that $(\partial\overline{p}/\partial y_*) = O(\epsilon^{\frac{3}{2}})$. Thus by matching with the outer region, the pressure is uniform and determined by the inner limit of $\overline{p}$ subject to an error $O(\epsilon^{\frac{3}{2}})$. With this for p and the particle-isentropic condition to determine the density, one obtains

$$(y/b) = t+1/(2t)+\epsilon[\overline{Y}_0+\overline{y}_0]_{\eta\to 0}+\epsilon^{\frac{3}{2}}[\overline{Y}_1+\overline{y}_1]_{\eta\to 0} \\ +\epsilon^{\frac{3}{2}}(2\ln Y_*)/\tilde{t}-\epsilon^{\frac{3}{2}}\tilde{t}^{-1}\,\text{F.P.}\int^{\infty}\dot{Y}_*^2 Y_* dY_*+O(\epsilon^2), \tag{6.6}$$

which matches with the outer-region solution.

In the inner region, the variable ζ is used instead of Y_* and the pressure gradient is exponentially small. With the particle-isentropic condition, and the boundary condition (2.3), (2.4) yields

$$\frac{y}{b} = t\left\{1+\frac{[2+\epsilon 4\ln 2+O(\epsilon^2)]\,\zeta}{2t[1+\epsilon^{\frac{1}{2}}\overline{p}_0(s,\tilde{t},0)+\epsilon\overline{p}_1(s,\tilde{t},0)+O(\epsilon^{\frac{3}{2}})]}\right\}^{\frac{1}{2}}. \tag{6.7}$$

Comparing the inner limit ($Y_*\to 0$) of (6.6) and the outer limit ($\zeta\to 1$) of (6.7), the matching between the intermediate and inner regions is accomplished provided

$$\left.\begin{aligned} [\overline{Y}_0+\overline{y}_0+\overline{p}_0/(2\tilde{t})]_{\eta\to 0} &= 0,\\ [\overline{Y}_1+\overline{y}_1+(\overline{p}_1-\overline{p}_0^2)/(2\tilde{t})]_{\eta\to 0} &= \frac{1}{\tilde{t}}\,\text{F.P.}\int_0^{\infty}\dot{Y}_*^2 Y_* dY_*-\frac{1}{8\tilde{t}^3},\end{aligned}\right\} \tag{6.8}$$

which corresponds to (4.20) in the plane case and may be identified with the pressure-volume relation of Cheng & Kirsch (1969). Equations (6.8) perform the role of inner boundary conditions to determine the functions Q, Q_1, A_1 and B_1.

Applying (6.3) and (6.4) to the first of (6.8) yields a linear integral-difference equation, which can be reduced to a D.D.E. comparable to (5.1a):

$$Q''(s,\tilde{t})-Q''(s-\tilde{t}^2/\chi_1,\tilde{t})+(2\tilde{t}/\Omega_1)\,[Q'(s,\tilde{t})+Q'(s-\tilde{t}^2/\chi_1,\tilde{t})] = 0. \tag{6.9}$$

The latter admits, quite similar to the plane case, a solution of the form

$$Q(s,t) = C_1(\tilde{t})\,e^{i\alpha_1 s}+D_1(\tilde{t}) \tag{6.10}$$

provided that
$$2\chi_1/\alpha_1 = \tan(\tfrac{1}{2}\alpha_1\Omega_1^2\chi_1). \tag{6.11}$$

Through the integral-difference equation, the functions A_1 and B_1 are related to the functions C_1 and D_1 as

$$A_1(\tilde{t}) = -(2/\Omega_1)\,D_1(\tilde{t}),\quad B_1(\tilde{t}) = \tilde{t}D_1(\tilde{t})-i(2/\alpha_1\Omega_1)\,C_1(\tilde{t}). \tag{6.12}$$

Parallel to the development in the plane case, (6.11) and the definitions of Ω_1 and χ_1 determine the transform function $\omega(t)$:

$$\omega_1(\tilde{t}) \equiv |\alpha_1|\omega(\tilde{t}) = \frac{1}{\tilde{t}}\int_0\frac{d(\tilde{t}^2)}{\mu(\tilde{t}^2)} = \frac{1}{\tilde{t}}\left\{\tan^{-1}\mu+\int_0\frac{\tan^{-1}\mu\,d\mu}{\mu}\right\} \tag{6.13}$$

where $\mu(\xi)$ is the same function given by (5.5) with $n = 0$.

From the second of (6.8), inner boundary conditions for the pressure in precisely the same form as (5.7a) can be obtained, into which the results of (6.5) and (6.10) are substituted, to yield

$$Q_1''(s,\chi_1)-Q_1''(s-\Omega_1^2\chi_1,\chi_1)+2\chi_1[Q_1'(s,\chi_1)+Q_1'(s-\Omega_1^2\chi_1,\chi_1)] \\ = E(\chi_1)+e^{i\alpha_1 s}\{L(\chi_1)\,(dC_1/d\chi_1)+M(\chi_1)\,C_1+N(\chi_1)\,C_1D_1\}+K(\chi_1)\,C_1^2e^{i2\alpha_1 s}, \tag{6.14}$$

where E, L, M, N and K are known functions of the slow variable χ_1. The right-hand side of (6.14) has three types of terms according to their functional dependence of s. The first type is independent of s and induces a particular solution for Q_1 proportional to s. The second type is proportional to $e^{i\alpha_1 s}$ and leads to the resonant solution for Q_1. The third type arises due to the non-linear effect and leads to an oscillatory solution proportional to $e^{i2\alpha_1 s}$. In order to keep Q_1 bounded, we set both $E(\chi_1)$ and the sum inside bracket $\{\}$ equal to zero, leading to

$$\left.\begin{aligned} &D_1 = D_{10}(\Omega_1 \chi_1)^{\frac{1}{2}}, \\ &L(\chi_1)\,(dC_1/d\chi_1) + [M(\chi_1) + N(\chi_1)\,D_1]\,C_1 = 0. \end{aligned}\right\} \qquad (6.15)$$

Unlike the function D in (5.11) for the plane case, D_1 will influence the physical quantities through C_1 and hence must be completely determined. It can be shown, however, that the integration constant D_{10} is to be matched to a non-existent perturbation of the order $\epsilon^{\frac{5}{12}}$ in the incipient-transition period; therefore, D_1 must be taken to be zero identically.

The determination of C_1, i.e. the second of integration of (6.15) and the matching with the large-time behaviour of the finite-time solution (2.7), follows that in the plane case. The solution yields the surface pressure in the principal transition period

$$[(p\tau^2)/(\rho_\infty b^2)]_{\eta\to 0} = 1 - \epsilon^{\frac{1}{2}} 4A_0\, e^{-\kappa_1(\mu)}[\mu^2(1+\mu^2)]^{-\frac{1}{2}} \cos[\omega_1(\tilde{t})\,t + \phi_0] + O(\epsilon), \quad (6.16)$$

where $$\kappa_1(\mu) = \frac{1}{4}\int_0 \left\{\frac{[(3-\mu^2)\tan^{-1}\mu - \mu]\,d\mu}{(1+\mu^2)\tan^{-1}\mu[\mu + (1+\mu^2)\tan^{-1}\mu]} - \frac{d\mu}{\mu}\right\},$$

which, like the plane case, contains the incipient-transition solution (2.8a) (to the first perturbation) as can be shown. Toward the end of this period, (6.16) gives

$$[(p\tau^2)/(\rho_\infty b^2)]_{\eta\to 0} = 1 - \epsilon^{\frac{1}{2}} 4A_0 (2\tilde{t}^2/\pi)^{-\frac{7}{4}} \cos\left[\frac{\pi}{\epsilon^{\frac{1}{2}}}\ln\left(\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\tilde{t}\right) + \frac{\pi}{2\epsilon^{\frac{1}{2}}} + \phi_0\right] + O(\epsilon), \quad \tilde{t}\to\infty. \quad (6.17)$$

7. Discussion of final results

7.1. *Transition curve for surface pressure*

To illustrate the complete transition, we give in figures 2 and 3 the pressure on contact surfaces $y_c \propto t$ in planar and axisymmetric cases, which are computed from (5.18) and (6.16) for a value of $\epsilon = \frac{1}{7}$ (i.e. $\gamma = \frac{7}{5}$). The results of the earlier period are shown (in fine line) as $\hat{p}_c \equiv p_c/\rho_\infty \dot{Y}_c^2$ *vs.* t; to facilitate comparison, the transition solutions (in bold line) are presented in the same variables. The need for taking ϵ to a specific value follows from the two-time character of the solutions, which precludes the possibility of scaling out ϵ entirely from the computed results. It may be pointed out that the constants of integration for the transition solutions cannot be arbitrarily adjusted, but are determined by the constants A_0 and ϕ_0 of (2.7) associated with the earlier period.

The existence of an overlapping range for the two solutions is evident from either figure. Compared to the earlier period results, the oscillations in the transition solutions give much smaller overshoots and undershoots and approach the asymptotic limits much sooner, bearing out the terminal behaviours (5.21)

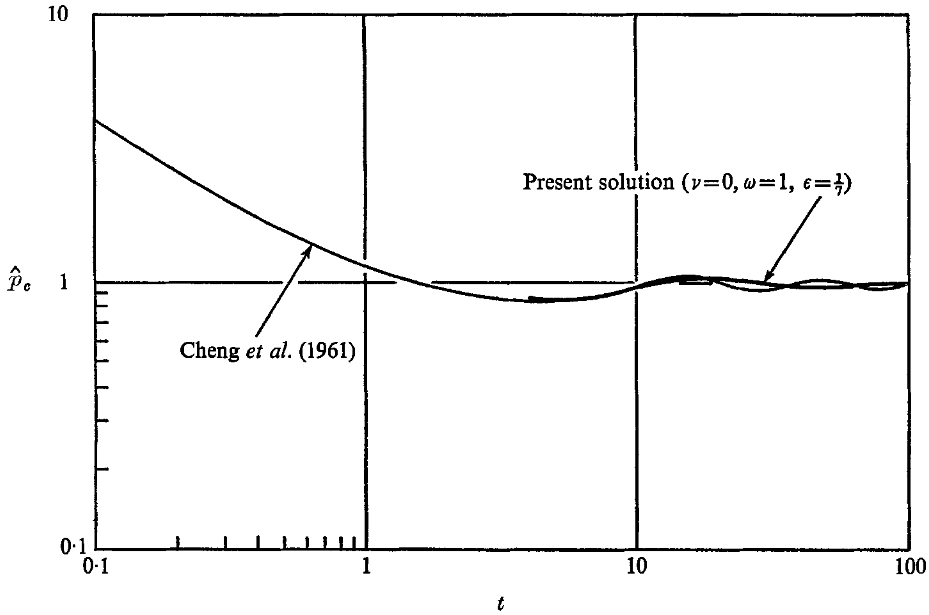

FIGURE 2. An example of the surface pressure history in the planar case of $y_c \propto t$ with $\epsilon \equiv (\gamma - 1)/2\gamma = \frac{1}{7}$. The variables $\hat{p}_c$ and t are dimensionless surface pressure and time, based on $\rho_\infty \dot{y}_c^2$ and $(\rho_\infty b^3/2\epsilon E_0)^{\frac{1}{2}}$, respectively.

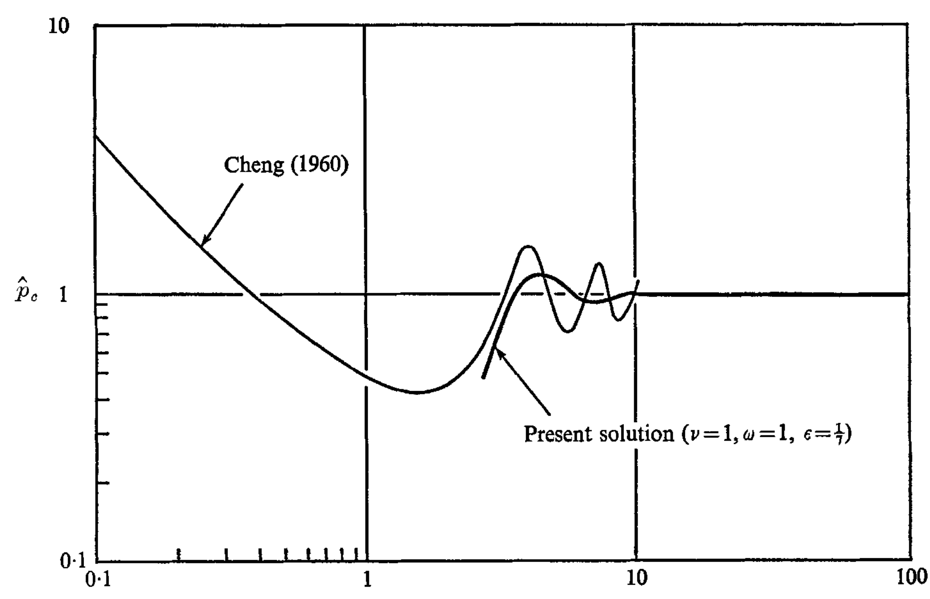

FIGURE 3. The surface pressure history in the cylindrical case of $y_c \propto t$ with $\epsilon \equiv (\gamma - 1)/2\gamma = \frac{1}{7}$. The variables $\hat{p}_c$ and t are dimensionless surface pressure and time, based on $\rho_\infty \dot{y}_c^2$ and $(\pi\rho_\infty b^4/2\epsilon E_0)^{\frac{1}{2}}$, respectively.

and (6.17). We note that corrections to the transition solutions in the next order, which are $O(\epsilon^{\frac14})$ for $\nu = 0$ and $O(\epsilon^{\frac12})$ for $\nu = 1$, are needed for a consistent match with the second-order solutions of Cheng & Kirsch (1969) and Kirsch (1969). A direct comparison of the present leading-order results with numerical characteristic solutions for $\epsilon = \frac17$ (Cleary & Axelson 1964; Cleary 1965; Henderson *et al.* 1966), is therefore not too meaningful at this stage and is omitted here.

7.2. *Comparison with a shock acoustic-wave interaction analysis*

The final approach given by (5.21) and (6.17), i.e. written in t, can be combined in a form which also applies to the spherical case

$$\hat{p}_c \sim 1 + \frac{A(\nu)}{\epsilon^{\frac54} t^{\frac12(4+3\nu)}} \cos\left[\frac{\pi(1+\nu)}{2\epsilon^{\frac12}} \ln t + \frac{K_\nu}{\epsilon^{\frac12}} + L_\nu + 0\right], \tag{7.1}$$

where $A(\nu)$, K_ν and L_ν are constants independent of ϵ. Thus, near the end of the transition, the oscillation period increases with t like $4\epsilon^{\frac12}t/(1+\nu)$ and the amplitude decays like $t^{-\frac12(4+3\nu)}$. The decrease in the period with the index ν shown above could have been inferred from an examination of the frequency of acoustic-wave reflexion between the shock and piston, ignoring the entropy wake. In the Newtonian limit, the distance between the strong shock and the driving piston $y_c \propto t$ is $\epsilon y_c/(1+\nu)$, it follows that the period for a complete reflexion of the acoustic wave is precisely $4\epsilon^{\frac12}t/(1+\nu)$.

In the absence of the entropy wake, i.e. without the initial explosion, the acoustic reflexion problem last mentioned is, of course, equivalent for the planar case, to the classical problem of shock/Mach-wave interaction in a supersonic wedge flow treated previously by Lighthill (1949), Chu (1952), and Chernyi (1959). Although not explicitly brought out by these authors, the problem admits, at large time, a family of oscillatory eigensolutions

$$p_c \propto t^{-A_n} \equiv t^{-a+ib_n}, \tag{7.2}$$

where A_n are complex exponents. This form was discovered by Ellinwood (1967) in his study of asymptotic hypersonic flows of blunted wedges and blunted cones; Ellinwood's basic model leads to inconsistent results for blunted cones and its full validity may be questioned, but the reduced equations solved are precisely those for the shock/acoustic-wave interaction (without the initial explosion) cited.† In the Newtonian limit, the exponent A_n of (7.2) can be obtained, for $y_c \propto t$, as

$$A_n = -\tfrac12(4+3\nu) + i\tfrac12\pi(1+2n)/\epsilon^{\frac12} \quad (n = 0, 1, 2, 3, 4, \ldots). \tag{7.3}$$

Thus the oscillatory decay of the present solution, (7.1), agrees with the large-time eigensolutions of the shock/acoustic-wave problem, but the comparison shows that only the *fundamental mode* of (7.3) is excited.‡ It should be pointed

† Large-time eigensolutions of the type (7.2) has also been studied recently by Stewartson & Thompson (1970) for a pure blast wave ($y_c = 0$).

‡ The second mode corresponding to $n = 1$ may be seen to tie up with the term p^3 appearing in (4.20), (6.2) and (6.8), which is seen to enter in the next approximation for the cylindrical case.

out that (7.2) and (7.3) give no preference to any particular mode, since the exponent of t in the amplitude is $-\frac{1}{2}(4+3\nu)$, independent of n. For this reason, (7.2) and (7.3) constitute no valid evidence of an oscillatory decay (inasmuch as a non-oscillatory function may submit to the Fourier analysis).

Although the validity of the present solution at large $\tilde{t}$ could be established through demonstrating regularity in the higher-order expansions, the foregoing discussion has left little doubt of its uniformity concerning the amplitude and frequency. It is interesting to note that, according to Schneider (1968), Chernyi's (1959) solution for $y_c \propto t$ in the planar case should behave at large t like $t^{-\frac{3}{2}} \cos(\ln t/\epsilon^{\frac{1}{2}} + \text{const.})$, which may be compared with $t^{-2} \cos(\pi \ln t/2\epsilon^{\frac{1}{2}} + \text{const.})$ from (7.1).

8. Concluding remarks

The foregoing analysis yields a description of the successive stages in which the shock layer produced by a point explosion reattaches to a driving piston; this, together with the earlier analyses (Cheng & Kirsch 1969), completes an approximate, but uniformly valid, dynamical picture of the problem studied. For the case $y_c \propto t$ analysed, the principal transition occurs in $\tilde{t} \equiv \epsilon^{1/(1+\nu)}t = O(1)$ where the solution has a three-deck construction. The analysis shows that the dynamics of the transition is dominated by an oscillation subject to modulations in both amplitude and frequency. In an interval of small $\tilde{t}$ corresponding to $1 \ll t \ll \epsilon^{-2/3(1+\nu)}$, the perturbation solution decays like $1/t^{\frac{1}{4}(3+\nu)}$ and oscillates with a frequency $(2\pi)^{-1}(1+\nu)\,t^{-\frac{1}{2}(1-\nu)}$, matching and confirming the earlier results of Cheng (1960) and Cheng *et al.* (1961). At a $\tilde{t}$ sufficiently large, the amplitude decays more rapidly as $\epsilon^{-\frac{5}{4}}t^{-\frac{1}{2}(4+3\nu)}$, and the frequency reduces further to $\frac{1}{4}\pi(1+\nu)\,\epsilon^{-\frac{1}{2}}t^{-1}$. This terminal behaviour is identified with the asymptotic form of a fundamental acoustic mode found in problems without explosion.

Unlike the analysis for the earlier period (Cheng & Kirsch 1969), the arena for the present analysis lies in the outer shock layer which, in the period under study, is found to be governed basically by an acoustic equation. The essential ingredient, which distinguishes the present analysis from a standard acoustic problem, are fed by the entropy wake. The wake imposes a singular, two-time character to the 'initial data', (2.7), and furnishes an inner boundary condition to the outer shock layer, as if it were a compliable surface. From the viewpoint of the Newtonian theory, the recovery of the acoustic equation is significant in revealing that the Busemann pressure formula tends to exaggerate the particle acceleration and is inadequate in describing certain important details of the reattachment.

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